Universality of the Inertial-Convective Range in Kraichnan's Model of a Passive Scalar

Gregory Eyink and Jack Xin
Department of Mathematics
University of Arizona
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We establish by exact, nonperturbative methods a universality for the correlation functions in Kraichnan's "rapid-change" model of a passively advected scalar field. We show that the solutions for separated points in the convective range of scales are unique and independent of the particular mechanism of the scalar dissipation. Any non-universal dependences therefore must arise from the large length-scale features. The main step in the proof is to show that solutions of the model equations are unique even in the idealized case of zero diffusivity, under a very modest regularity requirement (square-integrability). Within this regularity class the only zero-modes of the global many-body operators are shown to be trivial ones (i.e. constants). In a bounded domain of size L, with physical boundary conditions, the "ground-state energy" is strictly positive and scales as $L^{-\gamma}$ with an exponent $\gamma > 0$.

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The Kolmogorov 1941 theory (K41) [1] of fully developed turbulence hypothesized the existence of universal statistics at so-called *inertial-range* scales $L \gg \ell \gg \eta$, where L is the *integral length* characteristic of the peak energy and η is the *(Kolmogorov) microscale* characteristic of the peak dissipation. According to the first and second similarity hypotheses of Kolmogorov [1], both the limits $L \to \infty$ and $\eta \to 0$ should exist for the distribution function (PDF) of appropriate inertial-range variables, such as the velocity-differences $\delta v_\ell(x) = v(x+\ell) - v(x)$, and depend only on the length-scale ℓ and the mean dissipation ε per mass. However, much evidence has accumulated subsequently that there is a nontrivial dependence of such PDFs on the ratio L/ℓ [2]. This is associated to an increase in the fluctuations, or "intermittency," of the inertial-range variable for increasing ratio L/ℓ, so that the first limit $L \to \infty$ appears to lead to diverging moments. It has also been questioned whether the second limit $\eta \to 0$, physically associated to vanishing viscosity $\nu \to 0$, exists, or whether the limit is independent of the particular form of dissipation, such as hyperviscosity vs. ordinary viscosity. In a simple shell dynamical model, it was found by Leveque and She [3] that, while the limit $\nu \to 0$ appeared to exist, the values of the scaling exponents in the putative inertial range depended upon the degree of the hyperviscosity. Subsequent numerical study of Navier-Stokes turbulence [4] found no similar dependence, although the low Reynolds number of the simulation makes such a result preliminary. It may be taken as the current "conventional wisdom" that the limit $\eta \to 0$ exists and results in no ambiguity, while the limit $L \to \infty$ does not exist (at least for moments) and is a source of anomalous scaling, if not an outright breakdown of universality. However, the question which of the two limits exists, if either, and also the uniqueness of the limits must be regarded as open at the level of rigorous results. The issue frequently generates heated debates (e.g. [5]).

A simpler situation than the Navier-Stokes fluid for which many of the same questions arise is the problem of the turbulently convected passive scalar. The dynamics of the scalar field $\theta(\mathbf{r}, t)$ is given by the linear equation

$$(\partial_t + \mathbf{v}\cdot\boldsymbol{\nabla}_{\mathbf{r}})\theta = \kappa \triangle_{\mathbf{r}} \theta + f \tag{1}$$

in which $\mathbf{v}(\mathbf{r}, t)$ is the turbulent velocity and $f(\mathbf{r}, t)$ is a scalar source at macroscale L. The closest analogy exists for the so-called *inertial-convective range* of the passive scalar, in which the advecting velocities exhibit inertial-range scaling and the molecular diffusivity κ is small, so that the scalar dynamics itself is dominated by convection. In other words, this range corresponds to the idealized limit of infinite Péclet number, $\mathrm{Pe} = UL/\kappa = \infty$ (U is the typical velocity fluctuation at the macro length-scale). A dimensional theory for this situation analogous to that of Kolmogorov was developed soon after his by Obukhov and by Corrsin [6]. However, similar evidence has been found to suggest that the Obukhov-Corrsin theory is flawed in the same way as K41 [7]: while the existence of the $\kappa \to 0$ limit is not threatened by the present data, the experiments suggest that there is a nontrivial dependence upon L due to intermittent cascade of the scalar from the macroscale.

Recently, this "conventional picture" has obtained some support in a model of the passive scalar, first considered by Kraichnan in 1968 [8]. In this model, the (incompressible) velocity $\mathbf{v}$ and the source f are both taken to be Gaussian random fields delta-correlated in time. A regime mimicking the inertial-convective range of true turbulence is produced when the velocity covariance in space is taken to obey short-distance scaling:

$$\langle v_i(\mathbf{r}) v_j(\mathbf{0})\rangle \sim V_0\delta_{ij} - D\cdot r^{\zeta}\cdot\Big[\delta_{ij} + \frac{\zeta}{d-1}\left(\delta_{ij} - \frac{r_i r_j}{r^2}\right)\Big] \tag{2}$$

for $r \ll L$. Kraichnan's "rapid-change" model is exactly soluble in the sense that there is no closure problem, i.e.

the Nth correlator of the scalar, $\Theta_N(\mathbf{r}_1, ..., \mathbf{r}_N; t)$ obeys a linear equation

$$\partial_t \Theta_N = -\widehat{\mathcal{H}}_N \Theta_N + \sum_{\text{pairs } \{nm\}} F(\mathbf{r}_n, \mathbf{r}_m) \times \Theta_{N-2}(...\widehat{\mathbf{r}_n}...\widehat{\mathbf{r}_m}...), \tag{3}$$

which involves only itself and lower order correlators. The linear operator $\widehat{\mathcal{H}}_N$ is determined by the velocity statistics (see Eq.(4) below), while the inhomogeneous term involves the source covariance F. The new work has involved either (i) a physically-motivated ansatz for dissipative terms [9] or (ii) perturbative exploration of limiting regimes: $\zeta \ll 1$ in [10] and $\frac{1}{d} \ll 1$ in [11]. These studies all confirm the "conventional picture." Most recently [12], the $\frac{1}{d}$-expansion has predicted an interesting non-universality of the convective-range scaling exponents, through dependence on the correlation time of the random velocity field (when that time is sufficiently small). In addition, the above theories all make detailed quantitative predictions of convective range scaling exponents. Unfortunately, the two distinct approaches do not agree in detail and neither has complete control of errors. Numerical simulations [9,13] have not yet been able to probe the small ζ or large d ranges where conflicting predictions occur. Thus, the results, while exciting, are still unconfirmed.

We give a very direct, nonperturbative proof of the correctness of the "conventional picture" of inertial-convective range universality in the Kraichnan model. The full proofs are presented elsewhere [14], but the most significant details may be easily explained here. The main step in our argument is to show that the stationary solution of the model Eq.(3) is *unique* for the idealized $\kappa = 0$ case, when a certain regularity requirement is imposed. Our proof is inspired by an analogy of the model with the *quantum many-body problem.* In fact, the operator $\widehat{\mathcal{H}}_N$ for $\kappa = 0$ has the form of a quantum kinetic energy operator in Nd dimensions with a position-dependent *mass matrix* $\mathbf{M}(\mathbf{R})$, or $\widehat{\mathcal{H}}_N = (2\mathbf{M}(\mathbf{R}))^{-1} : \mathbf{PP}$. The mass matrix is defined in terms of the velocity covariance as

$$\left[\mathbf{M}(\mathbf{R})^{-1}\right]_{in,jm} = \langle v_i(\mathbf{r}_n) v_j(\mathbf{r}_m) \rangle, \tag{4}$$

where $\mathbf{R} = (\mathbf{r}_1, ..., \mathbf{r}_N)$ is the N-particle configuration point and $i, j = 1, ..., d,\ n, m = 1, ..., N$. $\mathbf{M}(\mathbf{R})$ is therefore a nonnegative matrix for every $\mathbf{R}$. Because of the quantum-mechanical analogy it is very natural to look for solutions among square-integrable functions ($\Theta_N \in \mathcal{L}^2$). However, the perturbation studies [11] suggest that the decay at large R is too slow for the solutions to be in $\mathcal{L}^2$. Since we are only interested anyway in the short-distance behavior, we confine the system to a bounded domain Ω in d-dimensions, to remove this difficulty. On the boundary of Ω we impose a fixed value T_0 of the scalar, as would be appropriate in a real experiment with a temperature field T held fixed to T_0 by thermal contact at the channel wall. Without loss of generality, we may take the boundary condition $\theta_0 = 0$ (Dirichlet b.c.), by considering the fluctuation field $\theta = T - T_0$ (as anticipated in Eq.(1).) We expect that our results will carry over also to more artificial geometries, e.g. the periodic b.c. used in numerical simulations [9], although some parts of our proof below depend upon the Dirichlet b.c.

There is a very simple formal argument which suggests the uniqueness of the $\kappa = 0$ solutions in $\mathcal{L}^2$. If we take G_N to be the inhomogeneous term in Eq.(3), then the stationary solution of the equation should be given uniquely by

$$\Theta_N = \widehat{\mathcal{H}}_N^{-1} G_N. \tag{5}$$

To permit this formal inversion, two things are required. First, it must be the case that $\widehat{\mathcal{H}}_N$ has no zero-modes. Secondly, it must also be shown that the range of $\widehat{\mathcal{H}}_N$ is closed (Fredholm condition). The insufficiency of the first condition alone may be explained more physically by observing that the inversion is still ill-defined if the operator $\widehat{\mathcal{H}}_N$ has no zero-modes but if 0 is a point of accumulation of the spectrum. In that case, the inverse $\widehat{\mathcal{H}}_N^{-1}$ is not a bounded operator and the righthand side of Eq.(5) need not exist. The Fredholm condition is guaranteed, in particular, if the "ground-state energy" of $\widehat{\mathcal{H}}_N$ is strictly positive. Unfortunately, there is a real danger that the condition is violated, because the mass matrix obviously has infinite mass eigenvalues at certain points $\mathbf{R}$. This certainly happens whenever two points coincide, $\mathbf{r}_n = \mathbf{r}_m$ for $n \neq m$, since then the velocity covariance is degenerate. Such large masses lead to the possibility of small, positive energies arbitrarily close to 0. Nevertheless, we have shown [14] not only that zero-modes do not exist with Dirichlet b.c. but also that the "ground-state energy" of $\widehat{\mathcal{H}}_N$ is then positive. We may note, incidentally, that a similar situation exists for periodic b.c. although (trivial) zero-modes certainly occur there, the constants. An equation may be developed for the *cumulant* part of the correlators, Θ_N^c, of the form $\widehat{\mathcal{H}}_N \Theta_N^c = G_N^c$, in which the inhomogeneous term G_N^c is defined in terms of lower-order cumulants and is explicitly orthogonal to constants in the $\mathcal{L}^2$ inner product. See Eq.(4.34) in [15]. One therefore has a similar formal solution $\Theta_N^c = \widehat{\mathcal{H}}_N^{-1} G_N^c$. As above, this is indeed the unique solution if (i) constants are the only zero-modes of $\widehat{\mathcal{H}}_N$ and (ii) there is a positive spectral gap above the 0 eigenvalue. We prove below that (i) holds, but (ii) is still open for periodic b.c.

Establishing these facts requires a careful study of the mass matrix, carried out in [14]. It is shown there that zero eigenvalues of the inverse (Gramian) matrix $\mathbf{G}(\mathbf{R}) = \mathbf{M}(\mathbf{R})^{-1}$ occur *only* when some points $\mathbf{r}_n, \mathbf{r}_m$ coincide, or are "fused." This result holds whenever the velocity spectrum in Fourier space is everywhere strictly positive. In general, the bad set of configurations $\mathbf{R}$ where zero

eigenvalues occur has K "clusters" of coinciding particles with N_k points in the kth cluster, $k = 1, ..., K$. It is shown in [14] that the precise number of zero eigenvalues at each "bad" point is $\sum_{k=1}^{K}(N_k - 1)d$. Furthermore, a lower bound is derived for the *least eigenvalue* $\nu(\mathbf{R})$ of $\mathbf{G}(\mathbf{R})$ valid for all $\mathbf{R}$, in terms of the minimum distance between any pair, $\rho(\mathbf{R}) = \min_{n\neq m} r_{nm}$. (Note that $r_{nm} = |\mathbf{r}_n - \mathbf{r}_m|$ is the distance between pairs $\mathbf{r}_n, \mathbf{r}_m$. With periodic b.c. r_{nm} should be taken to be the least distance between all of their periodic images). The estimate that is proved, for $0 < \zeta < 2$, is

$$\nu(\mathbf{R}) \geq (\text{const.})[\rho(\mathbf{R})]^{\zeta}. \tag{6}$$

Establishing this bound is the crucial part of the proof. Its validity is suggested by a simple perturbation theory calculation in $[\rho(\mathbf{R})]^{\zeta}$ using the asymptotic short-distance formula (2) for $\mathbf{G}(\mathbf{R})$. What is required is an argument that the first-order term in this expansion is non-vanishing. The difficulty is that there is a possible hierarchy of pair-separations $\{r_{nm}\}$ between the box size L and the minimum length $\rho(\mathbf{R})$. However, a careful examination of cases using the general first-order perturbation formula and the short-distance expression (2) show that the leading term is always the first-order one. This argument yields Eq.(6).

It is not hard to show from the above results for $\mathbf{M}(\mathbf{R})$ that constants are the only zero-modes of $\widehat{\mathcal{H}}_N$. In fact, the stochastic representation holds that $\langle\Theta_N, \widehat{\mathcal{H}}_N\Theta_N\rangle = \frac{1}{2}\int d\mathbf{R}\int d\mathbf{V} P_{\mathbf{R}}(\mathbf{V})|(\mathbf{V}\boldsymbol{\cdot}\boldsymbol{\nabla}_{\mathbf{R}})\Theta_N(\mathbf{R})|^2$, with $P_{\mathbf{R}}(\mathbf{V})$ the multivariate Gaussian with covariance $\mathbf{G}(\mathbf{R})$ for the Nd-vector $\mathbf{V}$. Because this probability density is strictly positive for every $\mathbf{R}, \mathbf{V}$ except for the "bad" points $\mathbf{R}$ of zero measure, it easily follows that any zero-mode satisfies $\boldsymbol{\nabla}_{\mathbf{R}}\Theta_N(\mathbf{R}) = \mathbf{0}$ almost everywhere. Hence, all zero-modes are constants. In the case of Dirichlet b.c. this rules out existence of any zero-modes at all.

The proof that the "ground-state energy" is strictly positive for Dirichlet b.c. uses the lower bound Eq.(6), giving

$$\langle\Theta_N, \widehat{\mathcal{H}}_N\Theta_N\rangle \geq (\text{const.})\int d\mathbf{R}\ [\rho(\mathbf{R})]^{\zeta}|\boldsymbol{\nabla}_{\mathbf{R}}\Theta_N(\mathbf{R})|^2 \tag{7}$$

For any function $g(\mathbf{R})$ whose Laplacian $\triangle_{\mathbf{R}}g$ is positive and finite except for a singularity manifold Γ of codimension ≥ 2, the inequality holds that

$$\int d\mathbf{R}\ |\triangle_{\mathbf{R}}\, g(\mathbf{R})||\varphi(\mathbf{R})|^2 \leq 4\int d\mathbf{R}\ |\triangle_{\mathbf{R}}\, g(\mathbf{R})|^{-1}|\boldsymbol{\nabla}_{\mathbf{R}}g(\mathbf{R})|^2|\boldsymbol{\nabla}_{\mathbf{R}}\varphi(\mathbf{R})|^2, \tag{8}$$

with $\varphi(\mathbf{R})$ any smooth function which $= 0$ on the boundary $\partial\Omega$. This is the result of an integration by parts and the Cauchy-Schwartz inequality [14,16]. It may be applied to $g(\mathbf{R}) = [\rho(\mathbf{R})]^{\zeta}$ since $\triangle_{\mathbf{R}}g(\mathbf{R}) = 2\zeta(d - \gamma)[\rho(\mathbf{R})]^{-\gamma}$ with $\gamma = 2 - \zeta$, which is positive as required when $\zeta > 0$ and $d \geq 2$. The Laplacian of $g(\mathbf{R})$ has singularities precisely on the "bad" set of points $\mathbf{R}$ with codimension $d \geq 2$ where mass eigenvalues become infinite. Thus, taking $g(\mathbf{R}) = [\rho(\mathbf{R})]^{\zeta}$ and $\varphi(\mathbf{R}) = \Theta_N(\mathbf{R})$ one obtains finally, if $\zeta > 0$,

$$\langle\Theta_N, \widehat{\mathcal{H}}_N\Theta_N\rangle \geq (\text{const.})\frac{(d-\gamma)^2}{2}\times \int d\mathbf{R}\ [\rho(\mathbf{R})]^{-\gamma}|\Theta_N(\mathbf{R})|^2 \tag{9}$$

for a set of $\Theta_N(\mathbf{R})$ dense in $\mathcal{L}^2$. Because $\rho(\mathbf{R}) \leq L$ everywhere, this implies that for $\gamma \geq 0$

$$\langle\Theta_N, \widehat{\mathcal{H}}_N\Theta_N\rangle \geq (\text{const.})\frac{(d-\gamma)^2}{2}L^{-\gamma}\|\Theta_N\|^2. \tag{10}$$

The last equation implies immediately a lower bound on the spectrum $\sim$ (const.)$L^{-\gamma}$ whenever $0 < \zeta < 2$ and $d \geq 2$. In fact, by a criterion of Friedrichs, it can be shown from the two estimates Eqs.(7) and (9) that the entire spectrum of $\widehat{\mathcal{H}}_N$ is discrete. See [16]. Thus, the solution at $\kappa = 0$ is given uniquely in $\mathcal{L}^2$ by Eq.(5) for Dirichlet b.c. Unfortunately, this proof does not work for periodic b.c. because the function $g(\mathbf{R})$ has additional singularities on the codimension-1 set of points where $r_{nm} = r_{n^*m}$, with $\mathbf{r}_{n^*}$ a periodic image of $\mathbf{r}_n$. These singularities contribute a surface term in the integration by parts argument which vitiates the derivation of the inequality Eq.(8). Despite the failure of this simple proof, we expect that there is still a gap in the spectrum above 0 for periodic b.c. and all our results remain true.

The proof of a unique solution in $\mathcal{L}^2$ for $\kappa > 0$ can be given along the same lines but is much easier, since there is then obviously a spectral gap of size $\sim \frac{\kappa}{L^2}$. This solution formally corresponds to a state of zero Prandtl number, $\text{Pr} = \nu/\kappa = 0$. At high-wavenumbers it exhibits an infinitely long "inertial-diffusive range" whose scalar spectrum decays with a large power. Such a range was proposed by Batchelor et al. in [17] and recently studied in the "rapid-change" model [18]. This range follows after a lower wavenumber inertial-convective range a lá Obukhov-Corrsin. Subsequently, one may consider the infinite Péclet number limit, in which the high wavenumber end of the inertial-convective range will move off to plus infinity. It is not hard to show that limits of the positive-κ solutions exist along subsequences for $\kappa \to 0$ and are $\mathcal{L}^2$ solutions of the $\kappa = 0$ equations [14]. Using the uniqueness of the $\kappa = 0$ solutions, it then follows that the $\kappa = 0$ solution is the limit of the positive-κ solutions. It therefore corresponds to an idealized inertial-convective range of infinite extent. The same argument applies to a broad class of dissipation terms besides the standard diffusion, including "hyperdiffusions" $\kappa_p(-\triangle_{\mathbf{r}})^p\theta$. The conclusion is obtained that, for all these dissipation mechanisms, the limits exist in $\mathcal{L}^2$ and coin-

cide with the unique $\kappa = 0$ solution (and therefore with each other). Universality is thus established.

There is a simple physical interpretation of our results which is worth emphasizing. Even in the limit as molecular diffusivity vanishes, $\kappa \to 0$, there is an effective diffusivity, or so-called *eddy-diffusivity*, acting on the mean scalar statistics which is generated through advection by the random velocity. For N-point statistics with $N \geq 2$ the effective diffusivity is scale-dependent, with $\kappa_{\rm eddy}(r) \sim D \cdot r^{\zeta}$ at length-scale r. This was first observed for 2-point statistics (pairs of Lagrangian tracers) by Richardson [19]. He postulated a diffusion equation of the same form as Eq.(3) with an eddy-diffusivity tensor $K_{ij}(r)$ scaling as Eq.(2) for $\zeta = \frac{4}{3}$. The matrix $\mathbf{G}(\mathbf{R})$ is the N-point generalization of the Richardson tensor and the Eq.(6) for its lowest eigenvalue is just an expression of the expected scaling for $\kappa_{\rm eddy}(r)$. Note that the L-dependence of the spectral gap for $\kappa = 0$ is a direct consequence, since it should plausibly scale as $\frac{\kappa_{\rm eddy}(L)}{L^2} \sim L^{-\gamma}$. It is important to observe that these eddy-diffusivity effects are only demonstrated for statistical correlations and not at the level of individual realizations of the scalar field. This explains the paradoxical-looking fact that as $\kappa \to 0$ the b.c. are satisfied by the statistical correlations, solving Eq.(3), but not necessarily by the individual scalar fields, with dynamics Eq.(1). Even if the scalar source f is removed, it may not be possible to maintain the Dirichlet b.c. on $\theta(\mathbf{r}, t)$ if $\kappa = 0$. For such ideal dynamics the scalar field (when smooth) is simply transported unchanged along Lagrangian characteristics. Any point $\mathbf{r}$ in the interior of Ω with initial $\theta_0(\mathbf{r}) \neq 0$ which subsequently flows into the boundary $\partial\Omega$ will violate the b.c. at that point. For $\kappa > 0$ the b.c. on θ are maintained by the action of the molecular diffusion on the strong gradients of scalar concentration formed by such advection to the boundary. This results in a thin "diffusive boundary layer", with thickness vanishing as $\kappa \to 0$, across which the scalar intensity drops rapidly to zero. For our model, the b.c. for statistical correlations are maintained in the same way by the eddy-diffusivity, whose action creates a "turbulent boundary layer" across which N-point functions drop to zero.

The conclusions obtained here hopefully extend to the true passive scalar and, even more optimistically, to the Navier-Stokes fluid velocity. It is, at least, satisfying to have one model in which the "conventional wisdom" for these turbulent systems may be verified without question. Our analysis has some further interest in terms of the perturbative studies [10,11]. Those works obtained nontrivial zero-modes of local approximations to $\widehat{\mathcal{H}}_N$ by means of a Rayleigh-Schrödinger perturbation theory. This does not contradict our result that all zero-modes are constants, because we consider the *global* operator. Our result just states that ambiguity in the perturbative treatments are removed by matching to the macro-scale L. In fact, our results confirm one of the basic assumptions used in the perturbative treatments: explicit matching to the dissipation scale may be replaced by a requirement of short-distance regularity. Even the weak requirement of local square-integrability is sufficient to select the solution properly matched to the dissipation range.

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[1] A. N. Kolmogorov, *C. R. Acad. Sci. URSS* **30** 301 (1941).
[2] A. Y. S. Kuo and S. Corrsin, J. Fluid. Mech. **50** 285 (1971); P. Kailasnath, K. R. Sreenivasan, and G. Stolovitsky, Phys. Rev. Lett. **68** 2766 (1992); A. Vincent and M. Meneguzzi, J. Fluid. Mech. **225** 1 (1991).
[3] E. Leveque and Z.-S. She, Phys. Rev. Lett. **75** 2690 (1995).
[4] N. Cao, S. Chen, and Z.-S. She, "Scalings and relative scalings in Navier-Stokes turbulence," Phys. Rev. Lett. (in press)
[5] T. Walstrom, "Turbulent diffusion phase transition is due to singular energy spectrum," preprint, LAUR-95-843 (1995).
[6] A. M. Obukhov. Izv. Akad. Nauk. SSSR, Geogr. i Geofiz. **13** 58 (1949); S. Corrsin, J. Appl. Phys. **22** 469 (1951).
[7] R. A. Antonia et al., Phys. Rev. **30** 2704 (1984).
[8] R. H. Kraichnan, Phys. Fluids **11** 945 (1968).
[9] R. H. Kraichnan, Phys. Rev. Lett. **72** 1016 (1994); R. H. Kraichnan, V. Yakhot, and S. Chen, **75** 240 (1995); A. L. Fairhall et al. Phys. Rev. E, in press (1996).
[10] K. Gawedzki and A. Kupiainen, Phys. Rev. Lett. **75** 3834 (1995); D. Bernard, K. Gawedzki and A. Kupiainen, preprint (1996), *chao-dyn/9601018*.
[11] M. Chertkov et al. Phys. Rev. E **52** 4924 (1995); M. Chertkov and G. Falkovich, submitted to Phys. Rev. Lett. (1996), *chao-dyn/9509007*.
[12] M. Chertkov, G. Falkovich, and V. Lebedev, submitted to Phys. Rev. Lett. (1996), *chao-dyn/9601016*.
[13] S. Chen and R. H. Kraichnan, "Randomly advected passive scalar: simulations and theory," preprint (1996).
[14] G. Eyink and J. Xin, submitted to Commun. Math. Phys.; *chao-dyn/9605008*, *mp_arc@ftp.ma.utexas.edu, #96-194*
[15] K. Gawedzki and A. Kupiainen, to appear in proceedings of the 34^{th} Schladming School of Nuclear Physics, *chao-dyn/9512006*.
[16] R. T. Lewis, Trans. Amer. Math. Soc. **271** 653 (1982).
[17] G. K. Batchelor, I. D. Howells, and A. A. Townsend, J. Fluid. Mech. **5** 134 (1959).
[18] U. Frisch and A. Wirth, "Batchelor inertial-diffusive range for a passive scalar advected by a white-in-time velocity field: spectrum and PDF," preprint (1996).
[19] L. F. Richardson, Proc. Roy. Soc. Lond. A **110** 709 (1926).